Multidual Quaternions and Higher-Order Analysis of Lower-Pair Kinematic Chains

D. Condurache[1], M. Cojocari[1], I. Popa[1]

[1]Department of Theoretical Mechanical
"Gheorghe Asachi" Technical University of Iasi
D. Mangeron Street no.59, 700050, Iasi,
Romania
daniel.condurache@tuiasi.ro

Abstract

This paper proposes a new computational method based on vector and quaternionic calculus and the properties of dual and multidual algebra for analysis of the higher-order acceleration field of spatial kinematics chains. First, a closed-form coordinate-free solution is presented, generated by the morphism between the Lie group of the rigid body displacements and the unit multidual quaternions. The solution is implemented for higher-order kinematics analysis of lower-pair serial chains. A general method for studying the vector field of arbitrary higher-order accelerations is described. The "automatic differentiation" feature of the multidual and hyper-multidual functions is used to obtain the higher-order derivative of a rigid body pose. This is obtained with no need for further differentiation of the body pose concerning time. It is proved that all information regarding the properties of the distribution of higher-order accelerations is contained in the specified unit hyper-multidual quaternion.

Keywords: Multidual algebra, Multidual quaternions, Higher-order kinematics, Lower-pair kinematic chains

1 Introduction

Studying the displacement and motion of rigid bodies is one of the principal issues in different research domains like robotics, theoretical kinematics, computer vision, astrodynamics, etc. A rigid body displacement is a transformation composed of a rotation and a translation. The key to the modern approach starts with the property of rigid body displacements group of forming a Lie group, accompanied by its Lie algebra. In modern terminology, the Lie group of rigid body displacement, SE (3), is the semidirect product of the rotation group $S\mathbb{O}_3$with the translation group. Also, the manifold is connected but not simply connected, and the manifold SE (3) is connected but not simply connected or compact. Recognizing the Lie group nature of rigid body motions and the Lie algebra nature of screws, all authors derived closed-form expressions of higher-order time derivatives of a twist [1]. Recent interest in explicit compact relations for higher-order time derivatives of twists (accelerations, jerk, jounce/snap, etc.) stems from advanced methods for the optimal trajectory planning and model-based control of robots and general multibody systems [1-2]. A previous result offers an isomorphism between the Lie group SE (3) with the group of the orthogonal dual tensors and Lie algebra**,** se (3)**,** of the Lie algebra of dual vectors (with vector product as an internal operation). The results obtained using dual algebras completely solve the problem of finding the field of higher-order accelerations utilizing the results obtained by the previous papers [3-7].

Moreover, the results can be extended for the multidual [7] and hyper-multidual commutative algebras [8]. This paper proposes a novel product of the exponential formula of hyper-multidual quaternions for studying the higher-order acceleration fields of rigid body motion of serial lower-pair kinematic chains using the calculus with multidual vectors and quaternions. The "automatic differentiation" feature of the multidual and hyper-multidual functions is used to obtain the higher-order derivative of a rigid body pose. This is obtained without requiring further differentiation of the body pose concerning time. Furthermore, it is proved that all information regarding the properties of the distribution of higher-order accelerations is contained in the specified unit hyper-multidual quaternion.

2 Higher-order kinematics of rigid body and instantaneous invariants

Let be a rigid motion given by a curve in Lie group of the rigid body displacements SE (3) given by the homogenous matrix $\mathbf{g}=\begin{bmatrix}\mathbf{R} & \boldsymbol{\rho}\\ \mathbf{0} & 1\end{bmatrix}$ where $\mathbf{R}\in S\mathbb{O}_3$ is a proper orthogonal tensor [3], [4],[6], $\mathbf{R}=\mathbf{R}(t)$, and $\boldsymbol{\rho}=\boldsymbol{\rho}(t)$ a vector functions of a time variable, n-time differentiable. As shown in [3], [4], [9], the n-th order acceleration of a point of the rigid body given by the position vector $\mathbf{r}$ in the fixed reference frame can be computed with the following equation [4]:

$$\mathbf{a}_{\mathbf{r}}^{[n]}=\mathbf{a}_n+\boldsymbol{\Phi}_n\mathbf{r};n\in\mathbb{N} \tag{1}$$

where the invariants tensor $\boldsymbol{\Phi}_n$ and the vector $\mathbf{a}_n$ is given by the below equations:

$$\boldsymbol{\Phi}_n=\mathbf{R}^{(n)}\mathbf{R}^T, \tag{2}$$

$$\mathbf{a}_n=\boldsymbol{\rho}^{(n)}-\boldsymbol{\Phi}_n\boldsymbol{\rho}. \tag{3}$$

Tensor $\boldsymbol{\Phi}_\mathrm{n}$ and vector $\mathbf{a}_\mathrm{n}$ generalize the notions of velocity / acceleration tensor respectively velocity / acceleration invariant. They are fundamental in the study of the vector field of the n^{th} order accelerations. The recursive formulas for computing $\boldsymbol{\Phi}_\mathrm{n}$ and $\mathbf{a}_\mathrm{n}$ are [4]:

$$\begin{cases}\boldsymbol{\Phi}_{n+1}=\dot{\boldsymbol{\Phi}}_n+\boldsymbol{\Phi}_n\boldsymbol{\Phi}_1\\ \mathbf{a}_{n+1}=\dot{\mathbf{a}}_n+\boldsymbol{\Phi}_n\mathbf{a}_1\end{cases},n\geq 1,$$
$$\boldsymbol{\Phi}_1=\tilde{\boldsymbol{\omega}},\mathbf{a}_1=\mathbf{v}_Q-\tilde{\boldsymbol{\omega}}\boldsymbol{\rho}_Q\triangleq\mathbf{v} \tag{4}$$

The pair of vectors $(\boldsymbol{\omega},\mathbf{v})$ is also known as *the spatial twist of rigid body*

In [3], [4], [9], an iterative procedure is described to determine the instantaneous tensor $\boldsymbol{\Phi}_n$, and vector $\mathbf{a}_n$ using the time derivative of spatial twist of rigid body motion, [4], [9]:

Theorem 1. [4] *There is a unique polynomial with the coefficients in the non-commutative ring of Euclidian second order tensors* $\mathbf{L}(\mathbb{V}_3,\mathbb{V}_3)$ *such that the vector respectively the tensor invariants of the* n^{th} *order accelerations will be written as:*

$$\begin{matrix}\mathbf{a}_n=\mathbf{P}_n(\mathbf{v})\\ \boldsymbol{\Phi}_n=\mathbf{P}_n(\tilde{\boldsymbol{\omega}})\end{matrix},n\in\mathbb{N}^* \tag{5}$$

where $\mathbf{P}_n$ *fulfills the relationship of recurrence:*

$$\begin{cases}\mathbf{P}_{n+1}=\mathbf{D}\mathbf{P}_n+\mathbf{P}_n(\tilde{\boldsymbol{\omega}}),n\in\mathbb{N}^*\\ \mathbf{P}_1=\mathbf{I}\end{cases} \tag{6}$$

with $\mathbf{D}=\frac{d}{dt}$ *the derivative operator with respect to time.*

For n=$\overline{1,4}$, it follows:

- *the velocity vector field invariants:*

$$\begin{cases}\mathbf{a}_1=\mathbf{v}\\ \boldsymbol{\Phi}_1=\tilde{\boldsymbol{\omega}}\end{cases}$$
$$\mathbf{a}_{\boldsymbol{\rho}}^{[1]}=\mathbf{v}+\tilde{\boldsymbol{\omega}}\boldsymbol{\rho} \tag{7}$$

- *the acceleration vector field invariants:*

$$\begin{cases}\mathbf{a}_2=\dot{\mathbf{v}}+\tilde{\boldsymbol{\omega}}\mathbf{v}\\ \boldsymbol{\Phi}_2=\dot{\tilde{\boldsymbol{\omega}}}+\tilde{\boldsymbol{\omega}}^2\end{cases}$$
$$\mathbf{a}_{\boldsymbol{\rho}}^{[2]}=\dot{\mathbf{v}}+\tilde{\boldsymbol{\omega}}\mathbf{v}+(\dot{\tilde{\boldsymbol{\omega}}}+\tilde{\boldsymbol{\omega}}^2)\boldsymbol{\rho} \tag{81}$$

- *the jerk vector field invariants:*

$$\begin{cases}\mathbf{a}_3 = \ddot{\mathbf{v}} + \tilde{\boldsymbol{\omega}}\mathbf{v} + 2\dot{\tilde{\boldsymbol{\omega}}}\mathbf{v} + \tilde{\boldsymbol{\omega}}^2\mathbf{v} \\ \boldsymbol{\Phi}_3 = \ddot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}\dot{\tilde{\boldsymbol{\omega}}} + 2\dot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}^3\end{cases} \tag{9}$$
$$\mathbf{a}_{\boldsymbol{\rho}}^{[3]} = \ddot{\mathbf{v}} + \tilde{\boldsymbol{\omega}}\dot{\mathbf{v}} + 2\dot{\tilde{\boldsymbol{\omega}}}\mathbf{v} + \tilde{\boldsymbol{\omega}}^2\mathbf{v} + (\ddot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}\dot{\tilde{\boldsymbol{\omega}}} + 2\dot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}^3)\boldsymbol{\rho}.$$

The higher-order time differentiation of spatial twist of rigid body motion solves completely the problem of determining the field of the higher- order acceleration of rigid motion [4-8]. Next, we present a new non-iterative procedure that permits the determination of the higher-order accelerations field using quaternions set in nilpotent algebra.

3 Mathematical preliminaries of multidual algebra, function, vectors, and quaternions

Let be $\underline{\mathbb{R}} = \mathbb{R} + \varepsilon_0\mathbb{R}; \varepsilon_0 \neq 0, {\varepsilon_0}^2 = 0$, the set of dual numbers, and $n \in \mathbb{N}, n \geq 1$, a natural number [3], [10]. We will introduce the set of hyper-multidual (HMD) numbers by: $\hat{\underline{\mathbb{R}}} = \underline{\mathbb{R}} + \varepsilon\underline{\mathbb{R}} + \cdots \varepsilon^n\underline{\mathbb{R}};\ \varepsilon \neq 0, \varepsilon^{n+1} = 0$. For n=1, on obtain hyper-dual numbers [9]. The se of multidual (MD) number is introduced by $\hat{\mathbb{R}} = \mathbb{R} + \varepsilon\mathbb{R} + \cdots + \varepsilon^n\mathbb{R}; \varepsilon \neq 0, \varepsilon^{n+1} = 0$. It can be easily proved that the set $\hat{\underline{\mathbb{R}}}$ with the addition operation and multiplication is a commutative ring with unit. An element from $\hat{\underline{\mathbb{R}}}$ is either invertible or zero divisor [7], [8]. The linear $\mathbb{R}$-algebra $\hat{\underline{\mathbb{R}}}$ is the direct product of dual algebra $\underline{\mathbb{R}}$ and multidual algebra $\hat{\mathbb{R}}$. $\hat{\underline{\mathbb{R}}}$ has a structure of 2(n+1) −dimensional associative, commutative, and unitary generalized Clifford Algebra, and $\hat{\underline{\mathbb{R}}} = \hat{\mathbb{R}} + \varepsilon_0\hat{\mathbb{R}};\ \varepsilon_0 \neq 0, {\varepsilon_0}^2 = 0$. $\hat{\mathbb{R}}$ is subalgebra of $\hat{\underline{\mathbb{R}}}$ by dimension n+1 over the real numbers field $\mathbb{R}$.

Let be $f: \underline{\mathbb{I}} \subseteq \underline{\mathbb{R}} \to \underline{\mathbb{R}}, f = f(\underline{x})$ a n-times differentiable dual function of dual variable [8]. We will define the HMD function of HMD variable $\hat{\underline{x}} = \underline{x} + \sum_{k=1}^{n} \underline{x}_k\varepsilon^k$, $f: \hat{\underline{\mathbb{I}}} \subseteq \hat{\underline{\mathbb{R}}} \to \hat{\underline{\mathbb{R}}}, f = f(\hat{\underline{x}})$ by equation:

$$f(\hat{\underline{x}}) = f(\underline{x}) + \sum_{k=1}^{n} \frac{\underline{\Delta}^k(\hat{\underline{x}})}{k!} f^{(k)}(\underline{x}), \tag{10}$$

here we noted by $\underline{\Delta}(\hat{\underline{x}}) = \hat{\underline{x}} - \underline{x} = \sum_{k=1}^{n} \underline{x}_k\varepsilon^k$, and $(\underline{\Delta}(\hat{\underline{x}}))^p = 0; p \geq n+1$. $\underline{\Delta}(\hat{\underline{x}})$ is multidual part of HDM number $\hat{\underline{x}}$.

Using the Eq. (10), we will define the following MD functions of MD variable:

$$\sin\hat{\underline{x}} = \sin\underline{x} + \sum_{k=1}^{n} \frac{\underline{\Delta}^k(\hat{\underline{x}})}{k!} \sin\left(\underline{x} + k\frac{\pi}{2}\right), \tag{11}$$

$$\cos\hat{\underline{x}} = \cos\underline{x} + \sum_{k=1}^{n} \frac{\underline{\Delta}^k(\hat{\underline{x}})}{k!} \cos\left(\underline{x} + k\frac{\pi}{2}\right), \tag{12}$$

MD and HMD vectors and tensors was studied in previous papers [7], [8], [10-12].

3.1 HDM vectors

Let be $\underline{\mathbb{V}}_3 = \mathbb{V}_3 + \varepsilon_0\mathbb{V}_3; \varepsilon_0 \neq 0, {\varepsilon_0}^2 = 0$, the set of dual vectors from three-dimensional Euclidean space. We will introduce the set of dual vectors in nilpotent algebra by: $\hat{\underline{\mathbb{V}}}_3 = \underline{\mathbb{V}}_3 + \varepsilon\underline{\mathbb{V}}_3 + \cdots + \varepsilon^n\underline{\mathbb{V}}_3; \varepsilon \neq 0, \varepsilon^{n+1} = 0$ (for n=1, obtain hyper-dual vectors [5]). $\hat{\underline{\mathbb{V}}}_3$ will be denote the set of hyper-dual (HDM) vectors.

A generic vector from $\hat{\underline{\mathbb{V}}}_3$ will be written as below:

$$\begin{gathered}\hat{\underline{\mathbf{a}}} = \underline{\mathbf{a}} + \underline{\mathbf{a}}_1\varepsilon + \cdots + \underline{\mathbf{a}}_n\varepsilon^n; \underline{\mathbf{a}}, \underline{\mathbf{a}}_k \in \underline{\mathbb{V}}_3 \\ \hat{\underline{\mathbf{b}}} = \underline{\mathbf{b}} + \underline{\mathbf{b}}_1\varepsilon + \cdots + \underline{\mathbf{b}}_n\varepsilon^n; \underline{\mathbf{b}}, \underline{\mathbf{b}}_k \in \underline{\mathbb{V}}_3\end{gathered} \tag{13}$$

We will define the scalar product respectively the cross product of two vectors from $\hat{\underline{\mathbb{V}}}_3$ by:

$$\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = \sum_{k=0}^{n} \sum_{p=0}^{k} \left(\underline{\mathbf{a}}_p \cdot \underline{\mathbf{b}}_{k-p} \right) \varepsilon^k \tag{14}$$

$$\hat{\mathbf{a}} \times \hat{\mathbf{b}} = \sum_{k=0}^{n} \sum_{p=0}^{k} \left(\underline{\mathbf{a}}_p \times \underline{\mathbf{b}}_{k-p} \right) \varepsilon^k \tag{15}$$

The triple vector product of three vectors $\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}$ is defined by $\langle \hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}} \rangle = \hat{\mathbf{a}} \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{c}})$.

All the three vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ represents a basis in the free module $\underline{\hat{\mathbb{V}}}_3$ if and only if $Re(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \neq 0$.

3.2 HDM Euclidean tensors

An $\hat{\mathbb{R}}$ - linear application of $\hat{\mathbb{V}}_3$ into $\hat{\mathbb{V}}_3$ is called a Euclidean tensor:

$$\hat{\mathbf{T}}(\hat{\lambda}_1 \hat{\mathbf{v}}_1 + \hat{\lambda}_2 \hat{\mathbf{v}}_2) = \hat{\lambda}_1 \hat{\mathbf{T}}(\hat{\mathbf{v}}_1) + \hat{\lambda}_2 \hat{\mathbf{T}}(\hat{\mathbf{v}}_2); \hat{\lambda}_1, \hat{\lambda}_2 \in \underline{\hat{\mathbb{R}}}, ; \forall \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2 \in \underline{\hat{\mathbb{V}}}_3. \tag{16}$$

Let $\mathbf{L}(\underline{\hat{\mathbb{V}}}_3, \underline{\hat{\mathbb{V}}}_3)$ be the set of tensors, then any $\hat{\mathbf{T}} \in \mathbf{L}(\underline{\hat{\mathbb{V}}}_3, \underline{\hat{\mathbb{V}}}_3)$, the transposed tensor denoted by $\hat{\mathbf{T}}^T$ is defined by $\hat{\mathbf{v}}_1 \cdot (\hat{\mathbf{T}} \hat{\mathbf{v}}_2) = \hat{\mathbf{v}}_2 \cdot (\hat{\mathbf{T}}^T \hat{\mathbf{v}}_1); \forall \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2 \in \underline{\hat{\mathbb{V}}}_3$.

While $\forall \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3 \in \underline{\hat{\mathbb{V}}}_3, Re(\langle \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3 \rangle) \neq 0$, the determinant is:

$$\langle \hat{\mathbf{T}} \hat{\mathbf{v}}_1, \hat{\mathbf{T}} \hat{\mathbf{v}}_2, \hat{\mathbf{T}} \hat{\mathbf{v}}_3 \rangle = \det \hat{\mathbf{T}} (\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3). \tag{17}$$

For any vector $\hat{\mathbf{a}} \in \hat{\mathbb{V}}_3$, the associated skew-symmetric tensor will be denoted by $\hat{\mathbf{a}} \times$ and will be defined $(\hat{\mathbf{a}} \times) \hat{\mathbf{b}} = \hat{\mathbf{a}} \times \hat{\mathbf{b}}, \forall \hat{\mathbf{b}} \in \underline{\hat{\mathbb{V}}}_3$.

The previous definition can be directly transposed into the following result: for any skew-symmetric tensor $\hat{\mathbf{A}} \in \mathbf{L}(\underline{\hat{\mathbb{V}}}_3, \underline{\hat{\mathbb{V}}}_3), \hat{\mathbf{A}} = -\hat{\mathbf{A}}^T$, a uniquely defined vector $\hat{\mathbf{a}} = vect(\hat{\mathbf{A}}), \hat{\mathbf{a}} \in \underline{\hat{\mathbb{V}}}_3$ exists in order to have $\hat{\mathbf{A}} \hat{\mathbf{b}} = \hat{\mathbf{a}} \times \hat{\mathbf{b}}, \forall \hat{\mathbf{b}} \in \underline{\hat{\mathbb{V}}}_3$. The set of skew-symmetric tensors is structured as a free $\hat{\mathbb{R}}$ - module of rank 3 and is isomorphic with $\underline{\hat{\mathbb{V}}}_3$.

3.3 HDM quaternions

A dual HMD quaternion can be defined as an associated pair of an HMD scalar quantity and a free HMD vector:

$$\underline{\hat{\mathbf{q}}} = (\underline{\hat{\mathrm{q}}}, \hat{\mathbf{q}}), \underline{\hat{\mathrm{q}}} \in \underline{\hat{\mathbb{R}}}, \hat{\mathbf{q}} \in \underline{\hat{\mathbb{V}}}_3, \tag{18}$$

The set of HMD quaternions will be denoted $\underline{\hat{\mathbf{Q}}}$ and is a $\underline{\hat{\mathbb{R}}}$-module of rank 4, if HMD quaternion addition and multiplication with HDM numbers are considered.

The product of two HDM quaternions $\underline{\hat{\mathbf{q}}}_1 = (\underline{\hat{\mathrm{q}}}_1, \hat{\mathbf{q}}_1)$ and $\underline{\hat{\mathbf{q}}}_2 = (\underline{\hat{\mathrm{q}}}_2, \hat{\mathbf{q}}_2)$ is defined by

$$\underline{\hat{\mathbf{q}}}_1 \underline{\hat{\mathbf{q}}}_2 = (\underline{\hat{\mathrm{q}}}_1 \cdot \underline{\hat{\mathrm{q}}}_2 - \hat{\mathbf{q}}_1 \cdot \hat{\mathbf{q}}_2, \underline{\hat{\mathrm{q}}}_1 \hat{\mathbf{q}}_2 + \underline{\hat{\mathrm{q}}}_2 \hat{\mathbf{q}}_1 + \hat{\mathbf{q}}_1 \times \hat{\mathbf{q}}_2). \tag{19}$$

Considering the above properties results that the $\underline{\hat{\mathbb{R}}}$-module $\underline{\hat{\mathbf{Q}}}$ becomes an associative, non-commutative linear dual algebra of order 4 over the ring of HMD numbers. For any HMD quaternion defined by Eq. (18), the followings can be computed: the conjugate denoted by $\underline{\hat{\mathbf{q}}}^* = (\underline{\hat{\mathrm{q}}}, -\hat{\mathbf{q}})$ and the norm denoted by $|\underline{\hat{\mathbf{q}}}|^2 = \underline{\hat{\mathbf{q}}} \underline{\hat{\mathbf{q}}}^*$. For $|\underline{\hat{\mathbf{q}}}| = 1$, any HDM quaternion is called unit HDM quaternion. Regarded solely as a free $\underline{\hat{\mathbb{R}}}$-module, $\underline{\hat{\mathbf{Q}}}$ contains two remarkable sub-modules: $\underline{\mathbf{Q}}_{\hat{\mathbb{R}}}$ and $\underline{\mathbf{Q}}_{\hat{\mathbb{V}}_3}$. The first one composed from pairs $(\underline{\hat{\mathrm{q}}}, \hat{\mathbf{0}}), \underline{\hat{\mathrm{q}}} \in \underline{\hat{\mathbb{R}}}$, isomorphic with $\underline{\hat{\mathbb{R}}}$, and the second one, containing the pairs $(\underline{0}, \hat{\mathbf{q}}), \hat{\mathbf{q}} \in \underline{\hat{\mathbb{V}}}_3$, isomorphic with $\underline{\hat{\mathbb{V}}}_3$.

Also, any HMD quaternion can be written as $\underline{\hat{\mathbf{q}}} = \underline{\hat{q}} + \hat{\mathbf{q}}$, where $\underline{\hat{q}} \triangleq (\underline{\hat{q}}, \underline{\mathbf{0}})$ and $\hat{\mathbf{q}} \triangleq (\underline{0}, \hat{\mathbf{q}},)$, or $\underline{\hat{\mathbf{q}}} = \hat{\mathbf{q}} + \varepsilon_0 \hat{\mathbf{q}}_0$, where $\hat{\mathbf{q}}$, $\hat{\mathbf{q}}_0$ are MD quaternions.

Let $\widehat{\mathbb{U}}$ denote the set of unit MD quaternions and $\underline{\widehat{\mathbb{U}}}$ denotes the set of units HDM quaternions. For any $\underline{\hat{\mathbf{q}}} \in \underline{\widehat{\mathbb{U}}}$, the following representation is valid:

$$\underline{\hat{\mathbf{q}}} = \left(1 + \varepsilon_0 \frac{1}{2} \hat{\boldsymbol{\rho}}\right) \hat{\mathbf{q}}\,, \tag{20}$$

where $\hat{\boldsymbol{\rho}} \in \widehat{\mathbb{V}}_3$ is MD vector and $\hat{\mathbf{q}} \in \widehat{\mathbb{U}}$ is unit MD quaternion. Also, a HDM number $\underline{\hat{\alpha}}$ and a unit HDM vector $\underline{\hat{\mathbf{u}}}$ exist so that:

$$\underline{\hat{\mathbf{q}}} = \cos\frac{\underline{\hat{\alpha}}}{2} + \underline{\hat{\mathbf{u}}} sin \frac{\underline{\hat{\alpha}}}{2} = \exp\left(\frac{\underline{\hat{\alpha}}}{2} \underline{\hat{\mathbf{u}}}\right), \tag{21}$$

where $\underline{\hat{\alpha}}$ and $\underline{\hat{\mathbf{u}}}$ are the natural HDM invariants of the rigid body motion.

Theorem 2. *The adjoint application*:

$$\mathbf{Ad}_{\underline{\hat{\mathbf{q}}}}: \underline{\widehat{\mathbb{V}}}_3 \to \underline{\widehat{\mathbb{V}}}_3, \qquad \mathbf{Ad}_{\underline{\hat{\mathbf{q}}}}(\,) = \underline{\hat{\mathbf{q}}}\,(\,)\underline{\hat{\mathbf{q}}}^* \tag{22}$$

is well definited, invertible, and have the properties:

$$\mathbf{Ad}_{\underline{\hat{\mathbf{q}}}}^{-1}(\,) = \underline{\hat{\mathbf{q}}}^*(\,)\underline{\hat{\mathbf{q}}} \tag{23}$$

$$\mathbf{Ad}_{\underline{\hat{\mathbf{q}}}_1\underline{\hat{\mathbf{q}}}_2} = \mathbf{Ad}_{\underline{\hat{\mathbf{q}}}_1}\mathbf{Ad}_{\underline{\hat{\mathbf{q}}}_2} \tag{24}$$

Remark 1. *Based on the construction of* $\underline{\widehat{\mathbb{U}}}$ *and the multiplication of dual quaternions, a direct conclusion is its Lie group structure* ($\underline{\widehat{\mathbb{V}}}_3$ *being the associated Lie algebra, with the cross product between* HMD *vectors as the internal operation*)*, which can be used to global parameterize all rigid motions*.

Using the internal structure of any element from $\underline{\widehat{S\mathbb{O}}}_3$ [8] the following theorem is valid:

Theorem 3.: *The Lie groups* $\underline{\widehat{\mathbb{U}}}$ *and* $\underline{\widehat{S\mathbb{O}}}_3$ *are linked by a surjective homomorphism:*

$$\Theta: \underline{\widehat{\mathbb{U}}} \to \underline{\widehat{S\mathbb{O}}}_3, \Theta\left(\underline{\hat{\mathbf{q}}}\right) = \underline{\mathbf{I}} + 2\underline{\hat{q}}\,(\hat{\mathbf{q}}\,\times) + 2(\hat{\mathbf{q}}\,\times)^2\,; \underline{\hat{\mathbf{q}}} = \underline{\hat{q}} + \hat{\mathbf{q}}\,. \tag{25}$$

Proof. In Eq. (25) denoted by $(\hat{\mathbf{q}}\,\times)$ the HMD skew-symmetric tensor [8] corresponding to the HDM vector $\hat{\mathbf{q}}$. Considering that any $\underline{\hat{\mathbf{q}}} \in \underline{\mathbb{U}}$ can be decomposed as in (21), results that $\Theta\left(\underline{\hat{\mathbf{q}}}\right) = exp\left(\underline{\hat{\alpha}}\,\underline{\hat{\mathbf{u}}}\,\times\right) \in \underline{\widehat{S\mathbb{O}}}_3$. This shows that the mapping given by (25) is well defined and surjective. Using direct calculus, we can also acknowledge that $\Theta\left(\underline{\hat{\mathbf{q}}}_2\underline{\hat{\mathbf{q}}}_1\right) = \Theta\left(\underline{\hat{\mathbf{q}}}_2\right)\Theta\left(\underline{\hat{\mathbf{q}}}_1\right)$. Regarding surjectivity, any orthogonal HMD tensor $\underline{\hat{\mathbf{R}}} \in \underline{\widehat{S\mathbb{O}}}_3$ can be represented as in [8], $\underline{\hat{\mathbf{R}}} = exp\left(\underline{\hat{\alpha}}\,\underline{\hat{\mathbf{u}}}\,\times\right)$. Thus, we can find a dual quaternion $\underline{\hat{\mathbf{q}}} = exp\left(\frac{\underline{\hat{\alpha}}}{2}\underline{\hat{\mathbf{u}}}\right)$ to have $\Theta\left(\underline{\hat{\mathbf{q}}}\right) = \underline{\hat{\mathbf{R}}}$, which proves that Θ is a surjective homomorphism. ■

4 Multidual differential transform and higher-order kinematics

So, being $f: I \subseteq \mathbb{R} \to \mathbb{R}, f = f(t)$, a real function of real time variable, n-th differentiable, $n \in \mathbb{N}$. To this function, it will be associated the multidual function of real variable $\check{f}$ given by the following equation:

$$\check{f} = f + \varepsilon\dot{f} + \cdots + \frac{\varepsilon^n}{n!} f^{(n)} = e^{\varepsilon\mathbf{D}} f, \tag{26}$$

where $e^{\varepsilon \mathbf{D}} = 1 + \varepsilon \mathbf{D} + \cdots + \frac{\varepsilon^n}{n!}\mathbf{D}^n$ with $\mathbf{D} = \frac{d}{dt}$ the derivative operator with respect to time.

Theorem 4. [7] *Being f and g two real function of class $C^n(I)$. The following properties take place*:

$$\widetilde{f+g} = \breve{f} + \breve{g}, \tag{27}$$

$$\widetilde{fg} = \breve{f}\breve{g}, \tag{28}$$

$$\widetilde{\lambda f} = \lambda \breve{f}, \forall \lambda \in \mathbb{R}, \tag{29}$$

$$\widetilde{f(\alpha)} = f(\breve{\alpha}), \alpha \in C^n(I), \tag{30}$$

$$\dot{\breve{f}} = \breve{\dot{f}}. \tag{31}$$

Let a rigid body motion parameterization by dual orthogonal tensor, [3], [4], $\underline{\mathbf{R}} = \underline{\mathbf{R}}(t) \in \underline{S\mathbb{O}}_3, \forall t \in I \subseteq \mathbb{R}$, can be defined by HMD tensor:

$$\underline{\breve{\mathbf{R}}} = e^{\varepsilon \mathbf{D}} \underline{\mathbf{R}}, \tag{32}$$

$$\underline{\breve{\mathbf{R}}} = \underline{\mathbf{I}} + (\sin \underline{\breve{\alpha}}) \underline{\breve{\mathbf{u}}} \times + (1 - \cos \underline{\breve{\alpha}}) (\underline{\breve{\mathbf{u}}} \times)^2 = \exp(\underline{\breve{\alpha}}\, \underline{\breve{\mathbf{u}}} \times), \tag{33}$$

where:

$$\underline{\breve{\alpha}} = e^{\varepsilon \mathbf{D}} \underline{\alpha}, \tag{34}$$

$$\underline{\breve{\mathbf{u}}} = e^{\varepsilon \mathbf{D}} \underline{\mathbf{u}}, \tag{35}$$

The higher-order acceleration field of rigid body motion are univocal determined by HMD orthogonal tensors [7]:

$$\underline{\breve{\mathbf{\Psi}}} = \underline{\breve{\mathbf{R}}}\, \underline{\mathbf{R}}^T = (\mathbf{I} + \varepsilon_0 \breve{\mathbf{a}} \times) \breve{\mathbf{\Phi}}, \tag{36}$$

where

$$\breve{\mathbf{\Phi}} = \mathbf{I} + \mathbf{\Phi}_1 \varepsilon + \cdots + \frac{\mathbf{\Phi}_n}{n!} \varepsilon^n, \tag{37}$$

$$\breve{\mathbf{a}} = \mathbf{a}_1 \varepsilon + \frac{\mathbf{a}_2}{2} \varepsilon^2 \ldots + \frac{\mathbf{a}_n}{n!} \varepsilon^n. \tag{38}$$

The n-th order acceleration of a point of the rigid body given by the position vector $\mathbf{r}$, denotes $\mathbf{a}_{\mathbf{r}}^{[n]}$, can be computed with the following relation [4-6], (see Eq.1):

$$\mathbf{a}_{\mathbf{r}}^{[n]} = \mathbf{a}_n + \mathbf{\Phi}_n \mathbf{r}; n \in \mathbb{N}, \mathrm{n} \geq 1. \tag{39}$$

In the case of helical rigid body motion ($\mathbf{u}$ = const., $\breve{\mathbf{u}} = \mathbf{u}$), from Eq. (33), (36), after some algebra, results that [6]:

$$\begin{cases} \breve{\mathbf{\Phi}} = \mathbf{I} + \sin \Delta\breve{\alpha}\, (\mathbf{u} \times) + (1 - \cos \Delta\breve{\alpha})(\mathbf{u} \times)^2 \\ \breve{\mathbf{a}} = \Delta\breve{\boldsymbol{\rho}} - \sin \Delta\breve{\alpha}\, \mathbf{u} \times \boldsymbol{\rho} - (1 - \cos \Delta\breve{\alpha}) \mathbf{u} \times (\mathbf{u} \times \boldsymbol{\rho})' \end{cases} \tag{40}$$

where with $\Delta\breve{x}$ was denoted the multidual part of the time function $\breve{x}$.

The calculations are considerably simplified by considering the rigid motion parametrized by the dual quaternion function: $\underline{\mathbf{q}} = \underline{\mathbf{q}}(t) \in \underline{\mathbf{U}}, \forall t \in \mathbb{I} \subseteq \mathbb{R}$. The relationship is easily demonstrated (see **Theorem 3**):

$$\underline{\breve{\mathbf{\Psi}}} = \Theta\left(\underline{\breve{\boldsymbol{\varphi}}}\right), \tag{41}$$

where $\underline{\breve{\boldsymbol{\varphi}}} = \underline{\breve{\mathbf{q}}}\,\underline{\mathbf{q}}^*$, $\underline{\mathbf{q}} = \exp(\frac{1}{2}\underline{\alpha}\,\underline{\mathbf{u}})$=$\cos\frac{1}{2}\underline{\alpha} + \underline{\mathbf{u}}\sin\frac{1}{2}\underline{\alpha}$, $\underline{\breve{\mathbf{q}}} = e^{\varepsilon\mathbf{D}}\underline{\mathbf{q}} = \cos\frac{1}{2}\underline{\breve{\alpha}} + \underline{\breve{\mathbf{u}}}\sin\frac{1}{2}\underline{\breve{\alpha}} = \exp(\frac{1}{2}\underline{\breve{\alpha}}\,\underline{\breve{\mathbf{u}}})$. In the case of helical rigid body motion ($\underline{\breve{\mathbf{u}}} = \underline{\mathbf{u}}$), $\underline{\breve{\boldsymbol{\varphi}}} = \underline{\breve{\mathbf{q}}}\,\underline{\mathbf{q}}^* = \exp(\frac{1}{2}\underline{\breve{\alpha}}\,\underline{\mathbf{u}})\exp(-\frac{1}{2}\underline{\alpha}\,\underline{\mathbf{u}}) = \exp\left(\frac{1}{2}\Delta\underline{\breve{\alpha}}\,\underline{\mathbf{u}}\right)$:

$$\underline{\breve{\boldsymbol{\varphi}}} == \exp\left(\frac{1}{2}\Delta\underline{\breve{\alpha}}\,\underline{\mathbf{u}}\right) = \cos(\frac{1}{2}\Delta\underline{\breve{\alpha}}) + \underline{\mathbf{u}}\sin(\frac{1}{2}\Delta\underline{\breve{\alpha}}). \tag{42}$$

The unique decomposition takes place:

$$\underline{\breve{\boldsymbol{\varphi}}} = \left(\mathbf{I} + \varepsilon_0\frac{1}{2}\breve{\mathbf{a}}\right)\breve{\boldsymbol{\varphi}}\,, \tag{43}$$

$$\breve{\boldsymbol{\Phi}} = \Theta(\breve{\boldsymbol{\varphi}}), \tag{44}$$

$$\breve{\mathbf{a}} = 2\frac{d}{d\varepsilon_0}\left(\underline{\breve{\boldsymbol{\varphi}}}\breve{\boldsymbol{\varphi}}^*\right), \tag{45}$$

In Eq. (45) will be denote $\frac{d}{d\varepsilon_0}\left(\breve{\mathbf{a}} + \varepsilon_0\breve{\mathbf{b}}\right) = \breve{\mathbf{b}}$.

5 Higher-order analysis of lower-pair kinematic chains

Consider a spatial kinematic chain of the rigid bodies $C_k, k = \overline{0,m}$ (fig.1). The relative motion of the rigid body C_k with respect to reference frame attached to C_{k-1} is described by the orthogonal dual unit quaternion ${}^{k-1}\underline{\mathbf{q}}_k$, $k = \overline{1,m}$.

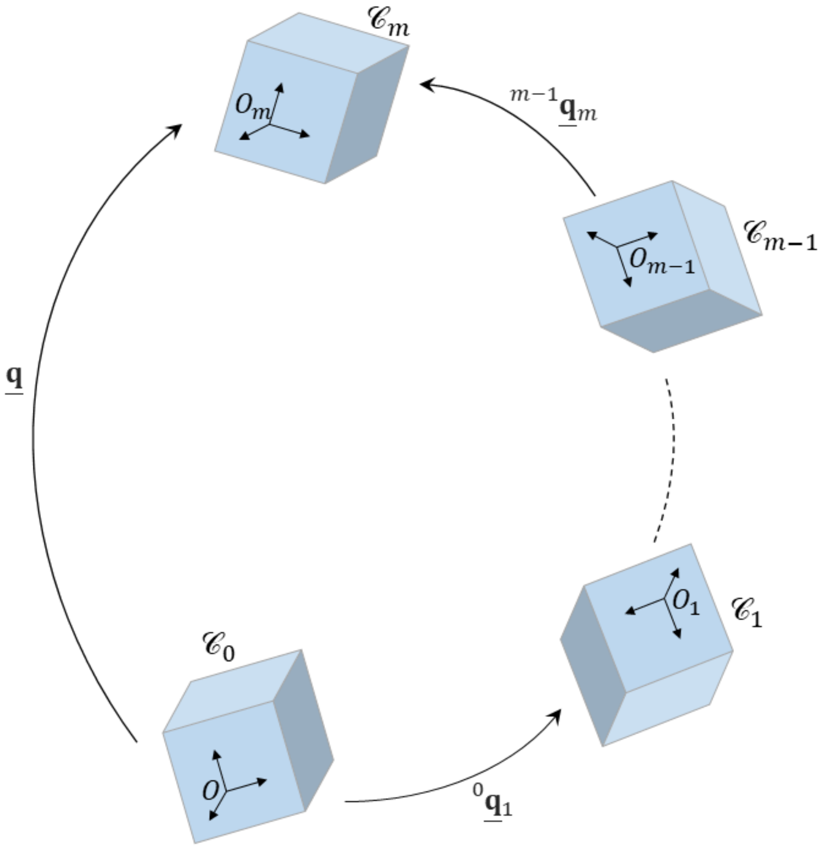

Fig.1: Relative motion properties of the terminal body C_m with respect to reference frame.

The relative motion properties of the terminal body C_m with respect to reference frame attached to C_0 (fig.1) are described by the unit dual quaternion [1], [2], [7]:

$$\underline{\mathbf{q}}_m = {}^0\underline{\mathbf{q}}_1\, {}^1\underline{\mathbf{q}}_2 \dots {}^{m-1}\underline{\mathbf{q}}_m, \tag{46}$$

$$\underline{\mathbf{q}}_m = \exp(\frac{1}{2}\underline{\alpha}_1\, {}^0\underline{\mathbf{u}}_1)\exp(\frac{1}{2}\underline{\alpha}_2\, {}^1\underline{\mathbf{u}}_2)\dots\exp(\frac{1}{2}\underline{\alpha}_m\, {}^{m-1}\underline{\mathbf{u}}_m) \tag{47}$$

If unit dual vectors ${}^{k-1}\underline{\mathbf{u}}_k$=const , $k = \overline{1,m}$, the spatial kinematic chain is named general mC manipulator. For any $\underline{\mathbf{q}} \in \underline{\mathbf{U}}$ on denote by $\mathrm{Ad}_{\underline{\mathbf{q}}}(\cdot) = \underline{\mathbf{q}}\,(\cdot)\,\hat{\underline{\mathbf{q}}}^*$ the adjoint map. The following theorem can be proved:

Theorem 5 *The vector fields of higher-order acceleration on terminal body of general mC manipulator given by the kinematic mapping (47) it results from HMD unit dual quaternion*:

$$\underline{\breve{\boldsymbol{\varphi}}}_m = \exp\left[\frac{1}{2}\underline{\mathbf{u}}_1\Delta\underline{\breve{\alpha}}_1\right]\exp\left[\frac{1}{2}\underline{\mathbf{u}}_2\Delta\underline{\breve{\alpha}}_2\right]\dots\exp\left[\frac{1}{2}\underline{\mathbf{u}}_m\Delta\underline{\breve{\alpha}}_m\right], \tag{48}$$

where $\underline{\mathbf{u}}_1 = {}^0\underline{\mathbf{u}}_1$, *and*:

$$\underline{\mathbf{u}}_k = \mathrm{Ad}_{{}^0\underline{\mathbf{q}}_1\,{}^1\underline{\mathbf{q}}_2 \dots {}^{k-2}\underline{\mathbf{q}}_{k-1}}\left({}^{k-1}\underline{\mathbf{u}}_k\right), k = \overline{2,m}. \tag{49}$$

are unit dual vectors corresponding to screw joint k, *and* $\Delta\underline{\breve{\alpha}}_k$ *denote the multidual part of HMD variable* $\underline{\breve{\alpha}}_k$, $k = \overline{1,m}$. *Mapping's* $\exp\,[\frac{1}{2}\underline{\mathbf{u}}_k\Delta\underline{\breve{\alpha}}_k]$, $k = \overline{1,m}$, *are polynomial and not transcendent, considering that* $(\Delta\underline{\breve{\alpha}}_k)^p = 0; p \geq n+1$.

Proof. Applying to the Eq. (47) the differential transform presented in **Theorem 4**, we will obtain:

$$\underline{\breve{\mathbf{q}}}_m = \exp\left[\frac{1}{2}\underline{\breve{\alpha}}_1\, {}^0\underline{\mathbf{u}}_1\right]\exp\left[\frac{1}{2}\underline{\breve{\alpha}}_2\, {}^1\underline{\mathbf{u}}_2\right]\dots\exp\,[\frac{1}{2}\underline{\breve{\alpha}}_m\, {}^{m-1}\underline{\mathbf{u}}_m]. \tag{50}$$

From Equation (46) result:

$$\underline{\mathbf{q}}_m^* = [\,{}^0\underline{\mathbf{q}}_1\, {}^1\underline{\mathbf{q}}_2 \dots {}^{m-1}\underline{\mathbf{q}}_m]^* \tag{51}$$

Considering that the unit HMD quaternion $\underline{\breve{\boldsymbol{\varphi}}}_m$ is given by equation:

$$\underline{\breve{\boldsymbol{\varphi}}}_m = \underline{\breve{\mathbf{q}}}_m\, \underline{\mathbf{q}}_m^* \tag{52}$$

From Eq. (52), Eq. (50), and Eq. (51) after some calculus and **Theorem 4** results Eq (48).■

Theorem 6 *The vector fields of higher-order acceleration on the terminal body in this body frame of the general mC manipulator, given by the kinematic mapping (47), results from HMD unit dual quaternion*:

$$\underline{\breve{\boldsymbol{\varphi}}}_m^B = \exp\left[\frac{1}{2}\underline{\mathbf{v}}_1\Delta\underline{\breve{\alpha}}_1\right]\exp\left[\frac{1}{2}\underline{\mathbf{v}}_2\Delta\underline{\breve{\alpha}}_2\right]\dots\exp\left[\frac{1}{2}\underline{\mathbf{v}}_m\Delta\underline{\breve{\alpha}}_m\right], \tag{53}$$

where unit dual vector $\underline{\mathbf{v}}_k$, $k = \overline{1,m}$:

$$\underline{\mathbf{v}}_k = \mathbf{Ad}_{[{}^{k-1}\underline{\mathbf{q}}_k\, {}^{k+1}\underline{\mathbf{q}}_{k+2} \dots {}^{m-1}\underline{\mathbf{q}}_m]^*}\left({}^{k-1}\underline{\mathbf{u}}_k\right). \tag{54}$$

are dual unit vectors corresponding to screw joint k, resolved in the body frame of C_m.

Proof. The higher-order acceleration field of terminal body expressed in the body frame attached to C_m are expressed by HMD quaternion:

$$\underline{\breve{\boldsymbol{\varphi}}}_m^B = \underline{\mathbf{q}}_m^*\underline{\breve{\mathbf{q}}}_m. \tag{55}$$

By Eq. (50), Eq. (51), and (55) after some algebra, on obtain:

$$\underline{\breve{\boldsymbol{\varphi}}}_m^B = \exp\left[\frac{1}{2}\underline{\mathbf{v}}_1\Delta\underline{\breve{\alpha}}_1\right]\exp\left[\frac{1}{2}\underline{\mathbf{v}}_2\Delta\underline{\breve{\alpha}}_2\right]\ldots\exp\left[\frac{1}{2}\underline{\mathbf{v}}_m\Delta\underline{\breve{\alpha}}_m\right], \tag{56}$$

where unit dual vector $\underline{\mathbf{v}}_k$, $k = \overline{1,m}$:

$$\underline{\mathbf{v}}_k = \mathbf{Ad}_{\left[{}^{k-1}\underline{\mathbf{q}}_k{}^{k+1}\underline{\mathbf{q}}_{k+2}\ldots{}^{m-1}\underline{\mathbf{q}}_m\right]^*}\left({}^{k-1}\underline{\mathbf{u}}_k\right). \blacksquare \tag{57}$$

The product of the exponential formula given by Eq. (53) and Eq. (56) contained all information regarding the properties of the distribution of higher-order accelerations for this serial lower-pair serial kinematic chain. If n=4, the velocity, acceleration, jerk, and jounce vector fields on the terminal body of the general mC manipulator will be simultaneously described.

6 Conclusions

A general method is proposed based on vector and quaternionic calculus and the properties of dual and multidual algebra to analyze the higher-order acceleration field of spatial kinematics chains. It is proved that all information regarding the properties of the distribution of higher-order accelerations field is encapsulated in the specified hyper-multidual quaternion. Furthermore, higher-order kinematics properties of lower-pair serial chains with nilpotent algebra are given with the product of the exponential formula. The results interest the theoretical kinematics, higher-order kinematics analysis of a serial manipulator, control theory, and multibody kinematics.

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